AXISYMMETRIC PROBLEM OF THERMOELASTICITY FOR A HOLLOW CYLINDER OF FINITE LENGTH

V. P. Merzlyakov

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The axisymmetric problem of heat conduction is examined for a hollow cylinder of finite length with mixed boundary conditions; a solution is given for the corresponding quasi-static problem of thermoelasticity.

We will consider the problem of the temperature field $t(r, z, \tau)$ of a hollow cylinder of length l with volume-distributed heat sources of intensity $q(r, z, \tau)$. Through the internal cavity of the cylinder flows a fluid whose temperature $\varphi(z, \tau)$ is a given function. Convective heat transfer takes place at the inner surface of the cylinder $r = R_1$; the outer surface $r = R_2$ is thermally insulated, and the temperature of the ends $z = 0$ and $z = l$ is equal to the fluid temperature. The initial temperature of the cylinder $f(r, z)$ is a function of the coordinates.

Using the substitution proposed in [1],

$$t = \varphi - \theta, \tag{1}$$

we transform the heat conduction equation and the boundary and initial conditions to the form

$$\frac{\partial \theta}{\partial \tau} = a \left\{ \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \theta}{\partial r}\right) + \frac{\partial^2 \theta}{\partial z^2}\right\} + \frac{\partial \varphi}{\partial \tau} - \frac{\partial^2 \varphi}{\partial z^2} - \frac{q}{c\gamma}, \tag{2}$$

$$\left.\frac{\partial \theta}{\partial r}\right|_{r=R_2} = 0, \quad \left.\frac{\partial \theta}{\partial r} - h_1 \theta\right|_{r=R_1} = 0, \tag{3}$$

$$\theta(r, 0, \tau) = 0, \ \theta(r, l, \tau) = 0, \tag{4}$$

$$\theta(r, z, 0) = \varphi(z, 0) - f(r, z). \tag{5}$$

Double application of the Grinberg method [2] gives the following solution of Eq. (2) with conditions (3)–(5):

$$\theta = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{4\mu_n^2}{R_1^2 B_n l}\, \theta_{nm} \sin \frac{m\pi z}{l}\, V_0\left(\mu_n \frac{r}{R_1}\right), \tag{6}$$

where

$$\theta_{nm} = \int_0^{\tau} \left[\frac{R_1^2 \mathrm{Bi}}{\mu_n^2} V_0(\mu_n) \times \int_0^{l} \left(\frac{\partial \varphi}{\partial \tau} - \frac{\partial^2 \varphi}{\partial z^2}\right) \sin \frac{m\pi z}{l}\, dz - \frac{1}{c\gamma} \int_0^{l} \sin \frac{m\pi z}{l}\, dz \times \int_{R_1}^{R_2} r q(r, z, \eta) V_0\left(\mu_n \frac{r}{R_1}\right) dr \right] \times \exp\left(\frac{a\mu_n \eta}{R_1^2} + \frac{am^2\pi^2\eta}{l^2}\right) \times \exp\left(-\frac{a\mu_n \tau}{R_1^2} - \frac{am^2\pi^2\tau}{l^2}\right) d\eta + \left[\frac{R_1^2 \mathrm{Bi}}{\mu_n^2} V_0(\mu_n) \int_0^{l} \varphi(z, 0) \sin \frac{m\pi z}{l}\, dz - \int_0^{l} \sin \frac{m\pi z}{l}\, dz \int_{R_1}^{R_2} r f(r, z) V_0\left(\mu_n \frac{r}{R_1}\right) dr \right] \times \exp\left(-\frac{a\mu_n^2 \tau}{R_1^2} - \frac{am^2\pi^2\tau}{l^2}\right); \tag{7}$$

$$V_0\left(\mu_n \frac{r}{R_1}\right) = Y_1(\mu_n k) J_0\left(\mu_n \frac{r}{R_1}\right) - J_1(\mu_n k) Y_0\left(\mu_n \frac{r}{R_1}\right). \tag{8}$$

Here, $V_0\left(\mu_n \frac{r}{R_1}\right)$ are the fundamental functions of the problem

$$\frac{d}{dr}\left(ar \frac{dV_0}{dr}\right) - \lambda r V_0 = 0,$$

$$V_0'\big|_{r=R_2} = 0,$$

$$V_0' - h_1 V_0\big|_{r=R_1} = 0;$$

μ_n are the roots of the transcendental equation

$$\frac{V_0(\mu_n)}{V_1(\mu_n)} = -\frac{\mu_n}{\mathrm{Bi}}, \tag{9}$$

$$V_1\left(\mu_n \frac{r}{R_1}\right) = Y_1(\mu_n k) J_1\left(\mu_n \frac{r}{R_1}\right) - J_1(\mu_n k) Y\left(\mu_n \frac{r}{R_1}\right), \tag{10}$$

$$B_n = \frac{4}{\pi^2} - V_0^2(\mu_n)(\mu_n^2 + \mathrm{Bi}^2). \tag{11}$$

From (1) we obtain

$$t = \varphi - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{4\mu_n^2}{R_1^2 B_n l}\, \theta_{nm} \sin \frac{m\pi z}{l}\, V_0\left(\mu_n \frac{r}{R_1}\right). \tag{12}$$

In solving the quasi-static problem of thermoelasticity by the Goodier method [3] the stresses are regarded as the sums of fictitious stresses of two types:

$$\sigma_{rr} = \bar{\sigma}_{rr} + \bar{\bar{\sigma}}_{rr}, \quad \sigma_{\varphi\varphi} = \bar{\sigma}_{\varphi\varphi} + \bar{\bar{\sigma}}_{\varphi\varphi},$$

$$\sigma_{zz} = \bar{\sigma}_{zz} + \bar{\bar{\sigma}}_{zz}, \quad \sigma_{rz} = \bar{\sigma}_{rz} + \bar{\bar{\sigma}}_{rz}. \tag{13}$$

The stresses of the first type have the form

$$\bar{\sigma}_{rr} = 2G\left(\frac{\partial^2\Phi}{\partial r^2} - \Delta\Phi\right),$$

$$\bar{\sigma}_{\varphi\varphi} = 2G\left(\frac{1}{r}\frac{\partial\Phi}{\partial r} - \Delta\Phi\right),$$

$$\bar{\sigma}_{zz} = 2G\left(\frac{\partial^2\Phi}{\partial z^2} - \Delta\Phi\right), \quad \bar{\sigma}_{rz} = 2G\frac{\partial^2\Phi}{\partial r\partial z}, \tag{14}$$

where Φ is the particular solution of the equation

$$\Delta\Phi = \frac{1+\nu}{1-\nu}\alpha t. \tag{15}$$

The stresses of the second type are expressed in terms of the Love biharmonic function

$$\bar{\bar{\sigma}}_{rr} = \frac{2G}{1-2\nu}\frac{\partial}{\partial z}\left(\nu\Delta - \frac{\partial^2}{\partial z^2}\right)L, \quad \bar{\bar{\sigma}}_{zz} =$$

$$= \frac{2G}{1-2\nu}\frac{\partial}{\partial z}\left[(2-\nu)\Delta - \frac{\partial^2}{\partial z^2}\right]L,$$

$$\bar{\bar{\sigma}}_{\varphi\varphi} = \frac{2G}{1-2\nu}\frac{\partial}{\partial z}\left(\nu\Delta - \frac{1}{r}\frac{\partial}{\partial r}\right)L, \quad \bar{\bar{\sigma}}_{zr} =$$

$$= \frac{2G}{1-2\nu}\frac{\partial}{\partial r}\left[(1-\nu)\Delta - \frac{\partial^2}{\partial z^2}\right]L. \tag{16}$$

The particular solution of (15) for temperature distribution (12) is

$$\Phi = \frac{1+\nu}{1-\nu}\alpha\int_0^z\left(\int_0^\eta \varphi(\xi,\ \tau)\,d\xi\right)d\eta +$$

$$+ \frac{1+\nu}{1-\nu}\alpha\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\sin\frac{m\pi z}{l}V_0\left(\mu_n\frac{r}{R_1}\right), \tag{17}$$

where

$$C_{nm} = \frac{4\mu_n^2\theta_{nm}}{R_1^2 B_n l(\mu_n^2/R_1^2 + m^2\pi^2/l^2)}. \tag{18}$$

Substituting (17) and (14), we obtain

$$\bar{\sigma}_{rr} = 2G\frac{1+\nu}{1-\nu}\alpha\left\{-\varphi(z,\ \tau) + \right.$$

$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\left[\frac{\mu_n}{R_1 r}V_1\left(\mu_n\frac{r}{R_1}\right) + \right.$$

$$\left.\left. + \frac{m^2\pi^2}{l^2}V_0\left(\mu_n\frac{r}{R_1}\right)\right]\sin\frac{m\pi z}{l}\right\},$$

$$\bar{\sigma}_{zz} = 2G\frac{1+\nu}{1-\nu}\alpha\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\frac{\mu_n^2}{R_1^2} \times$$

$$\times \sin\frac{m\pi z}{l}V_0\left(\mu_n\frac{r}{R_1}\right),$$

$$\bar{\sigma}_{\varphi\varphi} = 2G\frac{1+\nu}{1-\nu}\alpha\left\{-\varphi(z,\ \tau) + \right.$$

$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\left[\left(\mu_n^2/R_1^2 + \frac{m^2\pi^2}{l^2}\right)\times V_0\left(\mu_n\frac{r}{R_1}\right) - \right.$$

$$\left.\left. - (\mu_n/R_1 r)V_1\left(\mu_n\frac{r}{R_1}\right)\right]\sin\frac{m\pi z}{l}\right\},$$

$$\bar{\sigma}_{rz} = 2G\frac{1+\nu}{1-\nu}\alpha\times$$

$$\times\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\frac{\mu_n}{R_1}\frac{m\pi}{l}V_1\left(\mu_n\frac{r}{R_1}\right)\cos\frac{m\pi z}{l}. \tag{19}$$

We take the Love function in the form

$$L = \sum_{m=1}^{\infty}\left[a_m I_0\left(\frac{m\pi r}{l}\right) + b_m r I_1\left(\frac{m\pi r}{l}\right) + \right.$$

$$\left. + C_m K_0\left(\frac{m\pi r}{l}\right) + d_m r K_1\left(\frac{m\pi r}{l}\right)\right]\cos\frac{m\pi z}{l}. \tag{20}$$

Substituting (20) in (16), we obtain the following system of stresses:

$$\bar{\bar{\sigma}}_{rr} = \frac{2G}{1-2\nu}\sum_{m=1}^{\infty}\left\{a_m\left[-(1-2\nu)\frac{m^3\pi^3}{l^3}I_0 + \right.\right.$$

$$\left. + \frac{m^2\pi^2}{l^2 r}I_1\right] + (1-2\nu)b_m\frac{m^2\pi^2}{l^2}\left(I_0 - \frac{m\pi r}{l}I_1\right) +$$

$$+ c_m\left[-(1-2\nu)\frac{m^3\pi^3}{l^3}K_0 + \frac{m^2\pi^2}{l^2 r}K_1\right] +$$

$$\left. + (1-2\nu)d_m\frac{m^2\pi^2}{l^2}\left(K_0 - \frac{m\pi r}{l}K_1\right)\right\}\sin\frac{m\pi z}{l},$$

$$\bar{\bar{\sigma}}_{zz} = \frac{2G}{1-2\nu}\sum_{m=1}^{\infty}\left\{a_m(3-2\nu)\frac{m^3\pi^3}{l^3}I_0 + \right.$$

$$+ b_m\left[(3-2\nu)\frac{m^3\pi^3 r}{l^3}I_1 - \right.$$

$$\left. - 2(2-\nu)\frac{m^2\pi^2}{l^2}I_0\right] + c_m(3-2\nu)\frac{m^3\pi^3}{l^3}K_0 +$$

$$+ d_m\left[(3-2\nu)\frac{m^3\pi^3 r}{l^3}K_1 - \right.$$

$$\left.\left. - 2(2-\nu)\frac{m^2\pi^2}{l^2}K_0\right]\right\}\sin\frac{m\pi z}{l},$$

$$\bar{\bar{\sigma}}_{\varphi\varphi} = \frac{2G}{1-2\nu}\sum_{m=1}^{\infty}\left\{a_m\frac{m^2\pi^2}{l^2}\times\right.$$

$$\times\left[2\nu\frac{m\pi}{l}I_0 - \frac{1}{r}I_1\right] +$$

$$+ b_m\frac{m^2\pi^2}{l^2}\left[(1-2\nu)I_0 + 2\nu\frac{m\pi r}{l}I_1\right] + \tag{21}$$

$$+ c_m \frac{m^2\pi^2}{l^2}\left[2\nu \frac{m\pi}{l} K_0 - \frac{1}{r} K_1\right] +$$

$$+ d_m \frac{m^2\pi^2}{l^2}\left[(1-2\nu) K_0 +\right.$$

$$\left.\left. + 2\nu \frac{m\pi r}{l} K_1\right]\right\} \sin \frac{m\pi z}{l},$$

$$\bar{\bar{\sigma}}_{rz} = \frac{2G}{1-2\nu}\sum_{n=1}^{\infty}\left\{a_m (1-2\nu)\frac{m^3\pi^3}{l^3} I_1 - \right.$$

$$- b_m\left[2(1-\nu)\frac{m^2\pi^2}{l^2} I_1 + (1-2\nu)\frac{m^3\pi^3}{l^3} rI_0\right] +$$

$$+ c_m (1-2\nu)\frac{m^3\pi^3}{l^3} K_1 - d_m\left[2(1-\nu)\frac{m^2\pi^2}{l^2} K_1 +\right.$$

$$\left.\left. + (1-2\nu)\frac{m^3\pi^3}{l^3} rK_0\right]\right\} \cos \frac{m\pi z}{l}. \quad (21)\ \text{(cont'd)}$$

Satisfying the conditions

$$\bar{\sigma}_{rr} + \bar{\bar{\sigma}}_{rr} = 0,\ \bar{\sigma}_{rz} + \bar{\bar{\sigma}}_{rz} = 0 \ \text{ at } r = R_1 \text{ and } r = R_2,$$

we obtain a system of equations for a_m, b_m, c_m, and d_m:

$$\frac{(1+\nu)(1-2\nu)}{1-\nu}\alpha\left\{-\varphi_m + \right.$$

$$\left. + \frac{m^2\pi^2}{l^2}\sum_{m=1}^{\infty} C_{nm} V_0(\mu_n k)\right\} =$$

$$= a_m \frac{m^2\pi^2}{l^2}\left[\frac{1}{r} I_1 - (1-2\nu)\frac{m\pi}{l} I_0\right] +$$

$$+ (1-2\nu) b_m \frac{m^2\pi^2}{l^2}\left(I_0 - \frac{m\pi r}{l} I_1\right) +$$

$$+ c_m \frac{m^2\pi^2}{l^2}\left[\frac{1}{r} K_1 - (1-2\nu)\frac{m\pi}{l} K_0\right] +$$

$$+ (1-2\nu) d_m \frac{m^2\pi^2}{l^2}\left(K_0 - \frac{m\pi r}{l} K_1\right)\Big|_{r=R_2};$$

$$\frac{(1+\nu)(1-2\nu)}{1-\nu}\alpha\left\{-\varphi_m + \right.$$

$$\left. + \left(\frac{m^2\pi^2}{l^2} - \frac{\mathrm{Bi}}{R_1^2}\right)\sum_{n=1}^{\infty} C_{nm} V_0(\mu_n)\right\} =$$

$$= a_m \frac{m^2\pi^2}{l^2}\left[\frac{1}{r} I_1 - (1-2\nu)\frac{m\pi}{l} I_0\right] +$$

$$+ (1-2\nu) b_m \frac{m^2\pi^2}{l^2}\left(I_0 - \frac{m\pi r}{l} I_1\right) + c_m \frac{m^2\pi^2}{l^2} \times$$

$$\times \left[\frac{1}{r} K_1 - (1-2\nu)\frac{m\pi}{l} K_0\right] +$$

$$+ (1-2\nu) d_m \frac{m^2\pi^2}{l^2}\left(K_0 - \frac{m\pi r}{l} K_1\right)\Big|_{r=R_1};$$

$$0 = a_m (1-2\nu)\frac{m^3\pi^3}{l^3} I_1 - b_m \frac{m^2\pi^2}{l^2} \times$$

$$\times \left[2(1-\nu) I_1 + (1-2\nu)\frac{m\pi r}{l} I_0\right] +$$

$$+ c_m (1-2\nu)\frac{m^3\pi^3}{l^3} K_1 -$$

$$- d_m \frac{m^2\pi^2}{l^2}\left[2(1-\nu) K_1 + (1-2\nu)\frac{m\pi r}{l} K_0\right]\Big|_{r=R_2};$$

$$- \frac{(1+\nu)(1-2\nu)\alpha}{1-\nu}\ \frac{\mathrm{Bi}\, m\pi}{R_1 l}\sum_{n=1}^{\infty} C_{nm} V_0(\mu_n) =$$

$$= a_m (1-2\nu)\frac{m^3\pi^3}{l^3} I_1 - b_m \frac{m^2\pi^2}{l^2} \times$$

$$\times \left[2(1-\nu) I_1 + (1-2\nu)\frac{m\pi r}{l} I_0\right] +$$

$$+ c_m (1-2\nu)\frac{m^3\pi^3}{l^3} K_1 - d_m \frac{m^2\pi^2}{l^2}\left[2(1-\nu) K_1 +\right.$$

$$\left. + (1-2\nu)\frac{m\pi r}{l} K_0\right]\Big|_{r=R_1}.$$

It is easy to see that the conditions

$$\bar{\sigma}_{zz} + \bar{\bar{\sigma}}_{zz} = 0 \ \text{ at } \ z = 0 \text{ and } z = l$$

are satisfied.

Thus, stresses (13) satisfy zero boundary conditions at the surface of the cylinder, except for σ_{rz}, which generally speaking is nonzero at the ends. However, the σ_{rz} form an equilibrium stress system and, consequently, at a certain distance from the ends (13) is the solution of the problem posed.

NOTATION

$t(r, z, \tau)$ is the temperature of the cylinder, a function of the coordinates and time; $\varphi(z, \tau)$ is the fluid temperature, a given function; R_1 and R_2 are the inside and outside radii of cylinder: $k = R_2/R_1$; l is the length of the cylinder; a is the thermal diffusivity; c is the specific heat of the material; γ is the specific weight; λ is the thermal conductivity of material; α_1 is the heat transfer coefficient at the inner surface of the cylinder; $h_1 = \alpha_1/\lambda$; $B_i = \alpha_1 R_1/\lambda$ is the Biot number; μ_n is the root of the transcendental equation; Φ is the thermoelastic potential; L is the Love function; α is the coefficient of linear expansion; G is the shear modulus; ν is Poisson's ratio.

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